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THE SECOND FUNDAMENTAL FORM OF A PLANE FIELD

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This paper arose out of attempts to understand geometrically the meaning of various foliation invariants introduced in the last few years. Because these invariants are associated to the normal bundle, the characteristics of the normal plane field are important. Since the normal plane field is not integrable, one is led to study the Riemannian geometry of arbitrary plane fields, which is done by generalizing the second fundamental form. Concepts such as mean curvature and minimality can then be introduced for a plane field, and it can be shown that a totally geodesic plane field has vanishing second fundamental form. This is of interest because the normal plane field to an R-foliation is totally geodesic.

Given a foliation of a Riemannian manifold, a foliation connection is chosen in the normal bundle which is as compatible as possible with the Riemannian connection. Certain formulas are developed for the components of the connection and curvature forms, then used to prove a number of results, including: the leaf classes h_i for odd i depend only on the second fundamental form of the normal plane field, the Godbillon-Vey class in higher codimension is given by a formula analogous to that of Reinhart and Wood [8], and the reductions modulo the integers of certain leaf classes of dimension $4j - 1$ are independent of the choice of framing (they are defined only for framed foliations). A method for calculating the cohomology of truncated relative Weil algebras is essential to obtain the results. Such a method has been given in general by Kamber and Tondeur [9], [10], while more recently Guelorget and Joubert [5], building on their work, have given very explicit formulas for the case needed here, the general linear algebra modulo the orthogonal algebra.

Finally, some examples are given of vector fields in euclidean 3-space such that the normal plane fields are not integrable, and their second fundamental forms have certain prescribed properties.

1. Plane fields

A smooth p-plane field on a smooth Riemannian n-manifold is assumed. The inner product will be symbolized by $\langle\ ,\ \rangle$, while the Riemannian covari-

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ant derivative, connection form, and curvature form are denoted by ∇^*, θ^*, and Ω^*. $\{V_\alpha | \alpha = 1, \cdots, p\}$ is a local orthonormal basis for the plane field, $\{X_i | i = 1, \cdots, q = n - p\}$ is a local orthonormal basis for the normal plane field, and $\{\phi^\alpha, \omega^i\}$ is the corresponding coframe. U, V, W denote arbitrary tangent vectors of the frame and X, Y, Z arbitrary normal vectors belonging to the frame. The second fundamental form T of the plane field is defined by the formulas:

$$\langle T_V U, X\rangle = \tfrac{1}{2}\langle \nabla_V^* U + \nabla_U^* V, X\rangle , \qquad \langle T_V X, U\rangle = -\langle T_V U, X\rangle ,$$
$$T_X = 0 , \qquad \langle T_V U, W\rangle = \langle T_V X, Y\rangle = 0 .$$

The first and third formulas imply immediately that T is tensorial with respect to X, while the tensoriality with respect to U and V follows from

$$\langle T_V U, X\rangle = -\tfrac{1}{2}\{\langle U, \nabla_V^* X\rangle + \langle V, \nabla_U^* X\rangle\} .$$

In the case that the plane field is integrable, T is exactly the second fundamental form of the leaves as immersed submanifolds. In any case, it is a symmetric 2-form with values in the normal bundle. In a completely analogous way, we define the second fundamental form S of the normal plane field:

$$\langle S_X Y, U\rangle = \tfrac{1}{2}\langle \nabla_X^* Y + \nabla_Y^* X, U\rangle , \qquad \langle S_X U, Y\rangle = -\langle S_X Y, U\rangle ,$$
$$S_U = 0 , \qquad \langle S_X Y, Z\rangle = \langle S_X U, V\rangle = 0 .$$

Each of these tensors has a well-defined trace

$$\textstyle\sum_\alpha T_{V_\alpha} V_\alpha , \qquad \sum_i S_{X_i} X_i$$

which is a normal (respectively tangent) vector of the original plane field called the mean curvature vector. A plane field will be said to be minimal if the trace of its second fundamental form is the zero vector, and to be totally geodesic if each geodesic that is tangent to it at one point remains tangent for its entire length. The following proposition generalizes a well-known property of submanifolds.

Proposition 1. *If a plane field is totally geodesic, then its second fundamental form is identically* 0.

Proof. Since each unit vector U belonging to the plane field is the initial vector of a geodesic,

$$\langle T_U U, X\rangle = \langle \nabla_U^* U, X\rangle = 0$$

for every normal vector X. This implies $\langle T_U V, X\rangle = 0$ since $\langle T_U V, X\rangle = \langle T_V U, X\rangle$.

In particular, the hypotheses are satisfied by the normal plane field to a foliation with bundle-like metric (or R-foliations), so the second fundamental form of the normal plane field vanishes in this case.

In a way completely ananlogous to the definition of the second fundamental form, vector valued antisymmetric 2-forms B and A are defined. The equation

$$[X, Y] = \nabla^*_X Y - \nabla^*_Y X$$

shows that $A = 0$ if and only if the normal plane field is integrable. Similarly, $B = 0$ if and only if the given plane field is integrable. Hence these are known as the integrability tensors.

2. Foliations

For the remainder of the paper, it will be assumed that the given plane field is integrable, hence defines a foliation of codimension q. Then $B = 0$ and T is the usual second fundamental form of the leaves. Furthermore, there is a connection in the normal bundle which is well-adapted to the foliation and the metric [5] in the sence that

$$\nabla_X Y = \nabla^*_X Y\ , \quad \langle \nabla_V X, Y\rangle = \langle [V, X], Y\rangle\ , \quad \langle \nabla_V X, U\rangle = 0\ .$$

The corresponding connection and curvature forms will be denoted by θ and Ω. In terms of a local frame we have

$$\nabla_Y X_i = \textstyle\sum_j \theta_j{}^i(Y) X_j\ , \quad \nabla_V X_i = \sum_j \theta_j{}^i(V) X_j\ , \quad d\omega^i = \sum_j \omega^j \wedge \theta_j{}^i\ ,$$
$$\Omega_j{}^i = d\theta_j{}^i + \textstyle\sum_k \theta_k{}^i \wedge \theta_j{}^k\ , \quad \theta^*{}_j{}^i = -\theta^*{}_i{}^j\ , \quad \Omega^*{}_j{}^i = -\Omega^*{}_i{}^j\ .$$

The first two equations are the definition of $\theta_j{}^i$, the next two are proved by Guelorget and Joubert [5], and the last two are a restatement of the fact that the Lie algebra of the orthogonal group consists of skew-symmetric matrices. $\theta_j{}^i$ and $\Omega_j{}^i$ are neither symmetric nor skew-symmetric with respect to their indices, but useful formulas can be given for their symmetric and antisymmetric parts. Let $\theta_{Sj}{}^i = \frac{1}{2}(\theta_j{}^i + \theta_i{}^j)$ and $\theta_{Aj}{}^i = \frac{1}{2}(\theta_j{}^i - \theta_i{}^j)$ so that $\theta_j{}^i = \theta_{Sj}{}^i + \theta_{Aj}{}^i$. In a similar manner, $\Omega_j{}^i = \Omega_{Sj}{}^i + \Omega_{Aj}{}^k$.

Proposition 2.

$$\theta_j{}^i(X) = \theta_{Aj}{}^i(X) = \theta^*{}_j{}^i(X)\ ,$$
$$\theta_{Aj}{}^i(V) = \theta^*{}_j{}^i(V) + \langle V, A_{X_i} X_j\rangle\ ,$$
$$\theta_{Sj}{}^i(V) = \langle V, S_{X_i} X_j\rangle\ ,$$
$$\Omega_{Aj}{}^i = d\theta_{Aj}{}^i + \textstyle\sum_k \theta_{Ak}{}^i \wedge \theta_{Aj}{}^k + \sum_k \theta_{Sk}{}^i \wedge \theta_{Sj}{}^k$$
$$+ \tfrac{1}{2} \textstyle\sum_k (\theta_{Ak}{}^i \wedge \theta_{Sj}{}^k - \theta_{Sk}{}^i \wedge \theta_{Aj}{}^k)\ ,$$
$$\Omega_{Sj}{}^i = d\theta_{Sj}{}^i + \tfrac{1}{2}\textstyle\sum_k (\theta_{Ak}{}^i \wedge \theta_{Sj}{}^k + \theta_{Sk}{}^i \wedge \theta_{Aj}{}^k)\ .$$

Proof. The first formula is obvious, while the second and third formulas follow from

$$\begin{aligned}\theta_j{}^i(V) &= \langle \nabla_V X_i, X_j\rangle = \langle [V, X_i], X_j\rangle \\ &= \langle \nabla^*{}_V X_i, X_j\rangle - \langle \nabla^*{}_{X_i} V, X_j\rangle \\ &= \langle \nabla^*{}_V X_i, X_j\rangle + \langle V, \nabla^*{}_{X_i} X_j\rangle\end{aligned}$$

by decomposing into symmetric and antisymmetric parts. The last two formulas follow from decomposing the formula for $\Omega_j{}^i$ into symmetric and antisymmetric parts.

The property of θ which makes it interesting for study of characteristic classes is the Bott vanishing theorem [2], which is based on the following property of the curvature:

$$(1) \qquad \Omega_j{}^i(V_\alpha, V_\beta) = 0 \ .$$

3. Characteristic classes

The real valued characteristic classes of a vector bundle are represented in de Rham cohomology by products of differential forms c_i which locally can be written

$$c_i = \operatorname{tr}(\Omega^i) = \sum_{\alpha_1,\cdots,\alpha_i} \Omega_{\alpha_2}{}^{\alpha_1} \wedge \Omega_{\alpha_3}{}^{\alpha_2} \wedge \cdots \Omega_{\alpha_1}{}^{\alpha_i} ,$$

where $\Omega_\beta{}^\alpha$ is the curvature form of any connection in the bundle.

If i is odd, c_i is cohomologous to zero. Guelorget and Joubert [5] give the following local formula for global differential forms h_i, i odd, such that $dh_i = c_i$:

$$(2) \qquad h_i = i \operatorname{tr} \int_0^1 dt\theta_S\{t\Omega_S + \Omega_A + (t^2 - 1)\theta_S{}^2\}^{i-1} ,$$

where the multiplication of Lie algebra valued forms is interpreted in the same manner as in the formula for c_i. c_i is a scalar valued form of degree $2i$, while h_i is a scalar valued form of degree $2i - 1$. Formula (1) implies that for the normal bundle of a foliation of codimension q, any product of c_i's of total dimension greater than $2q$ is 0, hence that the characteristic ring vanishes above dimension $2q$ (see Bott [2]). The characteristic classes of the foliation are by definition the cohomology of the cochain complex consisting of products of the h_i (for i odd) and c_i (for all i) with the operation of exterior differentiation and the Bott vanishing relations [1], [3], [6]. In particular, the Godbillon-Vey class $h_1 c_1{}^q$ will be of interest in this paper. The formula (1) also implies that the restriction of h_i to any leaf is a closed form. This defines a ring of leaf classes on any leaf [7]. The aim of this section is to relate these classes to the geometry of the foliation by means of the Guelorget-Joubert formulas.

Lemma 1.

$$h_i = \sum_{\alpha+\beta+\gamma=i-1} B_{\alpha\beta\gamma} \operatorname{tr} (\Omega_S{}^\alpha \Omega_A{}^\beta \theta_S{}^{2\gamma+1}) ,$$

where

$$B_{\alpha\beta\gamma} = (-2)^\gamma \frac{(\alpha+\beta+\gamma+1)!}{\alpha!\,\beta!\,(\alpha+1)(\alpha+3)\cdots(\alpha+2\gamma+1)} .$$

Proof. Expand formula (2) by the multinomial formula to obtain coefficients

$$B_{\alpha\beta\gamma} = \frac{i!}{\alpha!\,\beta!\,\gamma!}\int_0^1 t^\alpha(t^2-1)^\gamma dt ,$$

which can then be evaluated by integration by parts.

In particular $h_1 = \operatorname{tr}(\theta_S)$ so that

$$\begin{aligned} &h_1(V) = \sum_i \theta_{Si}{}^i(V) = \sum_i \langle V, S_{X_i} X_i\rangle , \\ &h_1(X) = 0 . \end{aligned} \tag{3}$$

Thus h_1 is determined completely by the trace of the second fundamental form of the normal plane field, that is, its mean curvature vector. If the mean curvature vector is nonzero at a point, then in the neighborhood of that point it may be written as κV_1 where V_1 is a unit vector. Henceforth, it will be supposed that in any local basis $\{V_\alpha\}$ for the tangent plane to a foliation, V_1 is so chosen at any point where it is possible to do so. Then $h_1(V) = \kappa\langle V, V_1\rangle$.

In the case of h_i for i odd, $i > 1$, it is still possible to give a relatively easy formula for the value on an i-tuple of tangent vectors, since again only the pure θ_S term occurs. That is, we need to consider only the term

$$\frac{(-2)^{i-1} i!}{1\cdot 3\cdot 5\cdots(2i-1)} \operatorname{tr}(\theta_S{}^{2i-1}) , \tag{4}$$

which still depends only on the second fundamental form of the normal plane field.

Lemma 2. *If $k \neq 0$, then for $\alpha > 1$*

$$\begin{aligned} &c_1(V_\alpha, X_k) = -k\{\langle V_1, \nabla^*_{V_\alpha} X_k\rangle + \langle \nabla^*{}_{X_k} V_1, V_\alpha\rangle\} , \\ &(h_1 c_1{}^q)(V_1, V_{\alpha_1}, \cdots, V_{\alpha_q}, X_1, \cdots, X_q) \\ &\quad = \sum_\pi (-1)^\pi (-1)^{(q(q+1))/2} k^{q+1} \prod_{k=1}^q \{\langle V_1, T_{V_{\alpha_k}} X_k\rangle + \langle V_{\alpha_k}, \nabla^*{}_{X_k} V_1\rangle\} , \end{aligned}$$

where the summation is overall permutations of $\{\alpha_1, \cdots, \alpha_q\}$.

Proof.

$$c_1(V_\alpha, X_\kappa) = \sum_i d\theta_{S i}{}^i(V_\alpha, X_\kappa) = -\sum_i \theta_{S i}{}^i(\nabla^*{}_{V_\alpha}X_\kappa - \nabla^*{}_{X_\kappa}V_\alpha)$$
$$= -\kappa\langle V_1, \nabla^*{}_{V_\alpha}X_\kappa\rangle + \kappa\langle V_1, \nabla^*{}_{X_\kappa}V_\alpha\rangle ,$$

from which the first formula follows. Since $h_1(V_\alpha) = h_i(X_j) = 0$, the formula for $h_1c_1{}^q$ involves only

$$h_1(V_1)c_1(V_{\alpha_1}, X_1) \cdots c_1(V_{\alpha_q}, X_q) ,$$

which immediately implies the result.

The tensor $\langle \nabla^*{}_{X_k}V_1, V_\alpha\rangle = t(X_k, V_\alpha)$ Will be called the torsion of the normal plane field, by analogy to the codimension-1 case.

Theorem. *The form h_1 defining the first leaf class is the metric dual of the mean curvature vector of the normal plane field. More generally, the leaf classes h_i for odd i depend only on the second fundamental form of the normal plane field. The Godbillon-Vey class $h_1c_1{}^q$ depends on the mean curvature vector and the torsion tensor of the normal plane field, and on the second fundamental form of the leaves. If the normal plane field is minimal, h_1 and $h_1c_1{}^q$ vanish, while if it is totally geodesic, all the leaf and foliation classes vanish.*

Proof. The first statement follows from the formulas (3) and the fact that the mean curvature vector of the normal plane field is tangent to the leaves. The second statement follows from the fact that the leaf classes are defined by restriction of h_i to the leaves, so that formula (4) applies. At points where $\kappa \neq 0$, the Godbillon-Vey form is given by Lemma 2, while at points where $\kappa = 0$, $h_1(V) = 0$ for all V, so the Godbillon-Vey form vanishes. (Recall that $h_1(X)$ always vanishes.) This proves the third statement and the first part of the fourth statement. If the normal field is totally geodesic, $\theta_S = 0$, while every term in the formula of Lemma 1 contains a positive power of θ_S. This completes the proof of the theorem.

4. Foliations with trivial normal bundle

In general, the bases $\{V_\alpha, X_i\}$ and $\{\phi^\alpha, \omega^i\}$ can be constructed only locally, but the case where they exist globally is of special interest. The c_i are defined as before, but when the normal bundle is trivial, they are all cobundaries. In the classifying space for the general linear group, however, they are coboundaries only on the total space, not on the base space, so a universal formula for h_i, i even, with $dh_i = c_i$ can be given only on the principal bundle. By using a section, the h_i can be pulled back to forms on the base space, also denoted by h_i, but which depend on the choice of section. A formula for h_i is [5]

$$h_i = i \operatorname{tr} \int_0^1 dt\theta\{t\Omega + (t^2 - t)\theta^2\}^{i-1} ,$$

which by the same methods as in the proof of Lemma 1 is shown to be

$$h_i = \sum_{\alpha+\beta=i-1} A_{\alpha\beta} \operatorname{tr} (\Omega^2 \theta^{2\beta+1}) ,$$

where $A_{\alpha 0} = 1$ and for $\beta > 0$

$$A_{\alpha\beta} = (-)^\beta \frac{(\alpha + \beta)!\,(\alpha + \beta + 1)!}{\alpha!\,(\alpha + 2\beta + 1)!} .$$

The cochain complex consisting of cochains h_i and c_i, with the operation of exterior differentiation and the Bott vanishing relations gives rise to a set of characteristic classes which partially overlap those defined in the case of arbitrary normal bundle. For example, in codimension 2, in the general case the classes are c_2, $h_1c_1^2$, and h_1c_2 while in the case of trivial normal bundle, the classes are $h_1c_1^2$, h_1c_2, h_2c_2, $h_1h_2c_1^2$ and $h_1h_2c_2$. Also, the h_i for even i give rise to leaf classes by restricting to a leaf, just as they do for i odd. The object of this section is to obtain certain information about the way these classes depend on the geometry of the foliation and on the choice of frame.

Proposition 3. *The leaf class h_i for i even depends on the framing, the integrability tensor of the normal plane field, and the second fundmental form of the normal plane field. If the normal field is integrable and admits a Riemannian parallel framing, the leaf class is* 0. *If it admits a Riemannian parallel framing, the leaf class is independent of the choice among such framings.*

Proof. The only term in the formula for h_i which affects the leaf class is

$$-\frac{(i-1)!\,i!}{(2i-1)!} \operatorname{tr} \theta^{2i-1} .$$

On the other hand

$$\begin{aligned} &\sum_\alpha \theta_{S\alpha_2}{}^{\alpha_1} \theta_{S\alpha_3}{}^{\alpha_2} \wedge \cdots \wedge \theta_{S\alpha_1}{}^{\alpha_{2i-1}} \\ &\quad = \sum_\alpha \theta_{S\alpha_1}{}^{\alpha_2} \wedge \theta_{S\alpha_2}{}^{\alpha_3} \wedge \cdots \wedge \theta_{S\alpha_{2i-1}}{}^{\alpha_1} \\ &\quad = (-1)^{(i-1)(2i-1)} \sum_\alpha \theta_{S\alpha_{2i-1}}{}^{\alpha_1} \wedge \cdots \wedge \theta_{S\alpha_2}{}^{\alpha_3} \wedge \theta_{S\alpha_1}{}^{\alpha_2} = 0 , \end{aligned}$$

if i is even, since the last sum equals the first sum. Hence writing $\theta = \theta_A + \theta_S$ and expanding gives a formula in which every term contains a positive power of θ_A. The formulas for θ_A and θ_S given in the last section then imply the result.

Let κ be a real number such that κc_i is an integer class in the classifying space. Then Chern and Simons [4] have defined a cohomology class

$$(\kappa h_i)_1 \in H^{2i-1}(M; \boldsymbol{R}/\boldsymbol{Z})$$

on the base manifold M of any smooth vector bundle. This class is called a

Pontrjagin character. It is associated to κh_i in the sense that its pull back to the principal bundle is the reduction modulo Z of the real class defined by κh_i. Hence the restriction to any leaf of $(\kappa h_i)_1$ is the reduction modulo Z of the leaf class. Since $(\kappa h_i)_1$ does not depend on any choice of frame, neither does the reduction of the leaf class. Thus the following proposition has been proved.

Proposition 4. *The reducation modulo the integers of the leaf class h_i (i even) does not depend on the choice of framing, and is in fact the restriction to the leaf of the Pontrjagin character P_j ($j = i/2$) of the normal bundle.*

5. Examples

The simplest nonintegrable plane fields are normal p-plane fields to a vector field in euclidean three-space, but even in this case there are examples of totally geodesic plane fields, minimal fields which are not totally geodesic, and fields with nonzero principal curvatures of the same or of opposite sign.

Any Killing vector field is an R-foliation, hence gives $S = 0$. In the example

$$-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + \frac{\partial}{\partial z},$$

the straight lines normal to the orbits may be seen fairly easily. The nonzero components of A are given by

$$\pm(1 + x^2 + y^2)^{-1/2}(1 + x^2)^{-1/2}(1 + y^2)^{-1/2}.$$

A minimal plane field which is not totally geodesic is given by the normal field to

$$-x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

The principal curvatures (that is, the eigenvalues of S) are $\pm(2 + 2x^2)^{-1}$ and the nonzero components of A have the same values. The tangent vector fields to the lines of curvature (that is, the corresponding eigenvectors) are

$$\pm 2^{-1/2}\frac{\partial}{\partial x} + (2 + 2x^2)^{-1/2}\frac{\partial}{\partial y} + x(2 + 2x^2)^{-1/2}\frac{\partial}{\partial z}.$$

Perturbing slightly two-dimensional foliations by surfaces of positive (respectively negative) curvature will give examples of plane fields which are not integrable and which have principal curvatures of different absolute value and the same (respectively opposite) signs.

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